

# Variational principle for optimal accelerated neutralized flow

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Minimizing the energy deposited in the electron current in neutralized flows, such as in the Hall thruster, is examined. Modifying the electron current along the channel by inserting emitting electrodes, can enhance the efficiency. By employing variational methods, an optimal electron current distribution is found. The efficiency enhancement due to this effect, however, is shown to be small. © 2001 American Institute of Physics. [DOI: 10.1063/1.1331567]

## I. INTRODUCTION

Electric propulsion for space vehicles utilizes electric and magnetic fields to accelerate a propellant to a much higher velocity than chemical propulsion does, and, as a result, the required propellant mass is reduced. Among electric propulsion devices Hall thrusters offer much higher thrust density than conventional ion thrusters. Hall thrusters now perform with efficiencies of 50% in the important range of jet velocities of 15 000–25 000 m/s. Since the original ideas were introduced,<sup>1–7</sup> Hall thrusters have enjoyed both experimental and theoretical progress. References to the Hall thruster research can be found in a parallel publication.<sup>8</sup>

The Hall thruster employs a quasi-neutral plasma, and therefore, is not subject to a space-charge limit on the current. The accelerated ion flow is neutralized by electrons that flow in a direction opposite to the direction of the ion acceleration. The neutralization results in an inefficiency, since part of the power is deposited in the electron current. This wasted power is reduced in the Hall thruster by reducing the electron current, that is achieved by imposing a radial magnetic field along the axial drift of the electrons. The magnetic field impedes the electron mobility. However, it is advantageous to look for different means of minimizing the power wasted in the electron flow. In this paper we theoretically explore such different means. We relax the condition that it is the cathode only, located at the exit of the acceleration channel, that emits electrons. Rather, we search for an optimal distribution of emitting electrodes along the channel so that the wasted power is minimal. The theoretical result could be tested experimentally as a basis for an improved Hall thruster configuration. However, as we show here, the efficiency enhancement due to such emitting electrodes in the acceleration region is very small. The idea of adding electrodes fits into our ongoing research of a configuration of the Hall thruster, in which segmented side electrodes can either supply or absorb neutralizing electrons.<sup>9</sup> In our parallel publication<sup>8</sup> we have examined theoretically the case that an additional electrode absorbs electrons in the ionization zone, and have shown that it can significantly enhance the efficiency.

In Sec. II we present the model. In Sec. III we employ a variational method to find the optimal electron flow profile.

## II. THE MODEL

Let us assume that the ions in a plasma are accelerated by an imposed potential drop, while electrons acquire a velocity in the opposite direction by the electric field, a velocity that is proportional to their mobility. For simplicity, we neglect the processes of ionization and channel wall losses in this acceleration region and assume that all the ions have the same charge. If the plasma is quasi-neutral, the ion and the electron charge densities are equal

$$Z\Gamma_i \left[ \frac{m_i}{2Ze(\phi_A - \phi)} \right]^{1/2} = \frac{\Gamma_e}{\mu(x)(d\phi/dx)}. \quad (1)$$

Here  $\Gamma_i$  and  $\Gamma_e$  are the ion and electron particle fluxes,  $Ze$  and  $m_i$  are the ion charge and mass,  $\mu$  is the electron mobility,  $\phi$  is the potential, and  $\phi_A$  is the applied voltage. All quantities are assumed to depend on  $x$  only, the coordinate along the acceleration channel. The efficiency, the ratio of power absorbed by the accelerated ions to the total dissipated power by both ions and electrons, is

$$\eta = \frac{Z\Gamma_i \phi_A}{Z\Gamma_i \phi_A + \int_0^L dx \Gamma_e (d\phi/dx)}. \quad (2)$$

Equations (1) and (2) are, in dimensionless form

$$\Gamma_{eN} = \frac{1}{(1 - \psi)^{1/2}} \frac{d\psi}{d\zeta} \quad (3)$$

and

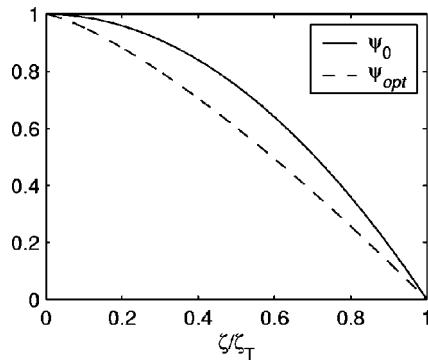
$$\eta = \frac{1}{1 + \int_0^{\zeta_T} d\zeta \Gamma_{eN} (d\psi/d\zeta)}. \quad (4)$$

Here

$$\zeta \equiv \left( \frac{2Ze}{m_i \phi_A} \right)^{1/2} \int_0^x \frac{dx'}{\mu(x')}, \quad (5)$$

and  $\zeta_T \equiv [2Ze/(m_i \phi_A)]^{1/2} \int_0^L dx / \mu(x)$ . Also  $\psi \equiv \phi / \phi_A$  and  $\Gamma_{eN} \equiv \Gamma_e / Z\Gamma_i$ .

In the regular diode, when there are neither sources nor sinks, the ion and electron fluxes are constant along the channel. In this case we solve Eq. (3) with the boundary conditions  $\Psi(0) = 1$  and  $\Psi(1) = 0$  and obtain that

FIG. 1. The profiles of  $\psi_0$  and of  $\psi_{opt}$  as functions of  $\zeta/\zeta_T$ .

$$\Gamma_{eN,0} = -\frac{2}{\zeta_T} \quad (6)$$

and that

$$\eta_0 = \frac{1}{1 - \Gamma_{eN,0}} = \frac{1}{1 + 2/\zeta_T}. \quad (7)$$

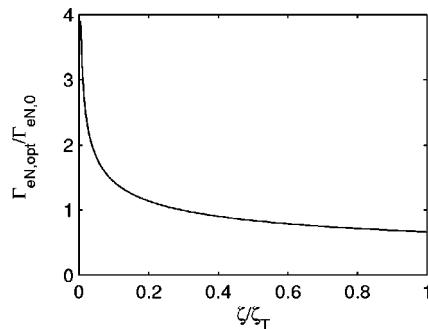
Also

$$\psi_0 = 1 - \left( \frac{\zeta}{\zeta_T} \right)^2. \quad (8)$$

### III. OPTIMAL ELECTRON FLOW PROFILE

We examine how modifying the profile of the electric potential can increase the acceleration efficiency. For such a modification to occur in a neutralized flow, the electron flow profile should be modified from being constant along the channel. Therefore, electrodes that act as electron sources or sinks should be added. In the Hall thruster the radial magnetic-field lines intersect the radial channel walls, so that locating electrodes along the walls and injecting or absorbing electrons along magnetic field lines is possible.

Let us entertain the possibility of having an arbitrary electric potential profile that is monotonically decreasing from the anode towards the cathode, for which the value of the normalized voltage between the anode and the cathode is held unity. The electron flux density  $\Gamma_{eN}$  is now not constant, but rather its value is determined locally by Eq. (3). We search the optimal potential distribution, in which  $\eta$  is maxi-

FIG. 2. The ratio  $\Gamma_{eN,0}/\Gamma_{eN,0}$  as a function of  $\zeta/\zeta_T$ .

mal. Thus, we look for the minimal value of the integral in the denominator of Eq. (4). We write this integral as

$$I = \int_0^{\zeta_T} d\zeta F(\psi, \psi'), \quad (9)$$

where

$$F(\psi, \psi') = \frac{(\psi')^2}{(1 - \psi)^{1/2}}, \quad (10)$$

and  $\psi' \equiv d\psi/d\zeta$ . We require that Euler's equation

$$\frac{d}{d\zeta} \left( \frac{\partial F}{\partial \psi'} \right) - \frac{\partial F}{\partial \psi} = 0, \quad (11)$$

be satisfied. The optimal electric potential profile, the solution of Eq. (11) that satisfies the boundary conditions, is

$$\psi_{opt} = 1 - \left( \frac{\zeta}{\zeta_T} \right)^{4/3}, \quad (12)$$

while the optimal electron flow profile is

$$\Gamma_{eN,0} = -\frac{4}{3} \frac{1}{\zeta_T} \left( \frac{\zeta_T}{\zeta} \right)^{1/3}. \quad (13)$$

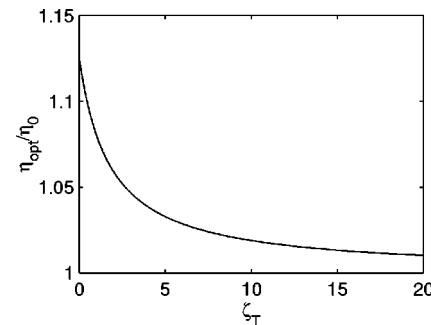
The efficiency of that optimal flow is found to be

$$\eta_{opt} = \frac{1}{1 + 16/(9\zeta_T)}. \quad (14)$$

Figure 1 shows the profiles of  $\psi_0$  and of  $\psi_{opt}$  and Fig. 2 shows  $\Gamma_{eN,0}/\Gamma_{eN,0}$ , both as functions of  $\zeta/\zeta_T$ . Figure 3 shows the ratio  $\eta_{opt}/\eta_0$  as a function of  $\zeta_T$ . As seen in Fig. 3, the efficiency enhancement is modest. The largest enhancement occurs for small  $\zeta_T$ , and it is 9/8.

We note that in practice one can add a limited number of electrodes along the channel only. If, for example, we add one electrode only, we can specify the normalized location  $\zeta_1$  of that electrode and the normalized potential  $\psi_1$  at which it is held. We denote the constant electron flow in the region  $\zeta_1 > \zeta > 0$  by  $\Gamma_1$  and the constant electron flow in the region  $\zeta_T > \zeta > \zeta_1$  by  $\Gamma_2$ . Employing Eq. (3), we derive the relations  $\Gamma_1 \zeta_1 = -2(1 - \psi_1)^{1/2}$  and  $\Gamma_2(\zeta_T - \zeta_1) = -2 + 2(1 - \psi_1)^{1/2}$ . With these expressions for the electron flows  $\Gamma_{1,2}$  the efficiency in this case  $\eta_1$  becomes

$$\eta_1 = \frac{1}{1 - \Gamma_1(1 - \psi_1) - \Gamma_2 \psi_1}. \quad (15)$$

FIG. 3. The ratio  $\eta_{opt}/\eta_0$  as a function of  $\zeta_T$ .

The maximal efficiency with respect to  $\psi_1$  and  $\zeta_1$  is found by solving  $\partial\eta_1/\partial\psi_1=0$  and  $\partial\eta_1/\partial\zeta_1=0$  for  $\psi_1$  and for  $\zeta_1$ . We find then that the values that yield the maximal efficiency are  $\psi_1=1-(\sqrt{2}-1)^2/9$  and  $\zeta_1/\zeta_T=(\sqrt{2}-1)^2/(2\sqrt{2}+1)$ . The efficiency in this case becomes

$$\eta_1 = \frac{1}{1 + 2(1 + 6\sqrt{2} + 16)/(27\zeta_T)}. \quad (16)$$

The maximal efficiency enhancement, when one electrode only is added, occurs also for small values of  $\zeta_T$ , and is 1.0594 only.

#### IV. CONCLUSIONS

Adding electron sources along the acceleration channel enables one to reduce the power that is wasted in heating the electrons and thus to increase the efficiency of the accelerator. We have shown here, though, that the efficiency increase due to this effect is small. Adding electrodes along the channel has other potential advantages, that are addressed elsewhere,<sup>8,9</sup> such as increasing the ionization and energy utilizations, and improving the plume collimation.

#### ACKNOWLEDGMENTS

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